Problem 4

Let $\{P_n\}$ be a sequence of points determined as in the figure. Thus $|AP_1| = 1$, $|P_nP_{n+1}| = 2^{n-1}$, and angle AP_nP_{n+1} is a right angle. Find $\lim_{n\to\infty} \angle P_nAP_{n+1}$.

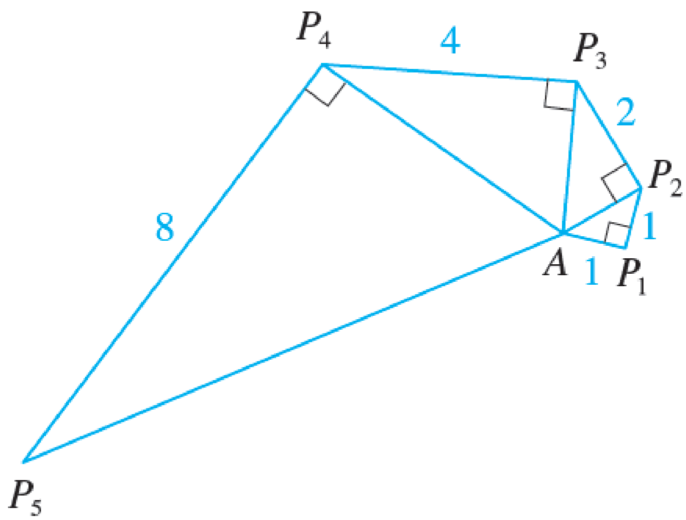

FIGURE FOR PROBLEM 4

Solution

The strategy for this problem is to assign a variable for the angle in question, write an equation for it in terms of n, and then to take the limit of it as n goes to infinity. Let $\theta_n = \angle P_nAP_{n+1}$ and let h_{n+1} be the hypotenuse of the triangle θ_n is in. These variables are illustrated below in the figures.

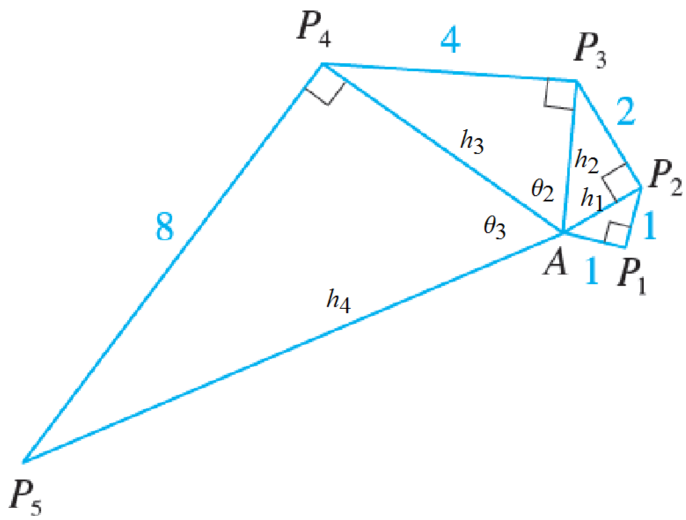

Figure 1: Problem figure with the hypotenuses and angles labeled.

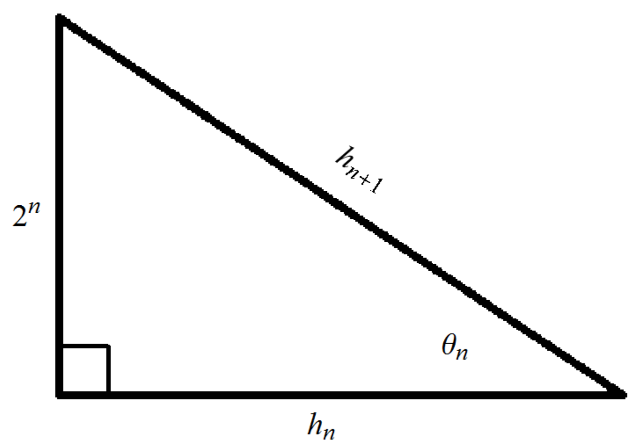

Figure 2: The nth triangle.

All these triangles have right angles, so trigonometric functions and the Pythagorean theorem can be applied. We'll use this equation to calculate θ_n.

$$\sin\theta_n = \frac{2^n}{h_{n+1}} \tag{1}$$

The objective now is to find an expression for h_{n+1}, the $(n+1)$th hypotenuse, in terms of n.

We'll start with finding h_1, then h_2, and try to find a pattern.

$$\begin{aligned} h_1^2 = 1^1 + 1^2 = 2 \quad &\rightarrow \quad h_1 = \sqrt{2} \\ h_2^2 = h_1^2 + 2^2 = 6 \quad &\rightarrow \quad h_2 = \sqrt{6} = \sqrt{2+4} \\ h_3^2 = h_2^2 + 4^2 = 22 \quad &\rightarrow \quad h_3 = \sqrt{22} = \sqrt{2+4+4^2} \\ h_4^2 = h_3^2 + 8^2 = 86 \quad &\rightarrow \quad h_4 = \sqrt{86} = \sqrt{2+4+4^2+4^3} \\ h_5^2 = h_4^2 + 16^2 = 342 \quad &\rightarrow \quad h_5 = \sqrt{342} = \sqrt{2+4+4^2+4^3+4^4} \end{aligned}$$

The general formula for the $(n+1)$th hypotenuse is

$$h_{n+1} = \sqrt{2 + \sum_{k=1}^{n} 4^k}.$$

The formula for the first n terms of a geometric series is the following.

$$\sum_{k=1}^{n} a_1 r^{k-1} = a_1 \cdot \frac{1-r^n}{1-r}$$

This means that

$$\sum_{k=1}^{n} 4^k = \sum_{k=1}^{n} 4 \cdot 4^{k-1} = 4 \cdot \frac{1-4^n}{1-4} = \frac{4}{3}(4^n - 1).$$

Thus, equation (1) becomes

$$\begin{aligned} \sin\theta_n &= \frac{2^n}{\sqrt{2 + \frac{4}{3}(4^n - 1)}} \\ &= \frac{1}{\frac{1}{2^n}\sqrt{2 + \frac{4}{3}(4^n - 1)}} \\ &= \frac{1}{\sqrt{\frac{1}{2^{2n}}\left[2 + \frac{4}{3}(4^n - 1)\right]}} \\ &= \frac{1}{\sqrt{\frac{2}{2^{2n}} + \frac{4}{3}\left(1 - \frac{1}{2^{2n}}\right)}}. \end{aligned}$$

Finally, take the limit of both sides as n goes to infinity.

$$\lim_{n\to\infty} \sin\theta_n = \lim_{n\to\infty} \frac{1}{\sqrt{\cancel{\frac{2}{2^{2n}}} + \frac{4}{3}\left(1 - \cancel{\frac{1}{2^{2n}}}\right)}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Therefore,

$$\lim_{n\to\infty} \theta_n = \frac{\pi}{3}.$$